Journal of Mathematical Analysis and Applications **243,** 373–408 (2000)
doi:10.1006/jmaa.1999.6677, available online at http://www.idealibrary.com on IDEAL®

Jacobi Approximations in Certain Hilbert Spaces and Their Applications to Singular Differential Equations

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Received July 30, 1998

Jacobi approximations in certain Hilbert spaces are investigated. Several weighted inverse inequalities and Poincaré inequalities are obtained. Some approximation results are given. Singular differential equations are approximated by using Jacobi polynomials. This method keeps the spectral accuracy. Some linear problems and a nonlinear logistic equation are considered. The stabilities and the convergences of proposed schemes are proved strictly. The main idea and techniques used in this paper are also applicable to other singular problems in multiple-dimensional spaces.

1. INTRODUCTION

The spectral method has high accuracy and often provides good numerical solutions of differential equations. But this merit might be destroyed by some facts. The first is the instability of nonlinear computations. The second is due to the discontinuities of data. The third is caused by the singularities of solutions. Some techniques have been proposed to overcome them. First, Kreiss and Oliger [1], Gottlieb and Turkel [2], Kuo [3], Vandeven [4], Tadmor [5], and Guo [6] provided various filterings to weaken the instability in nonlinear computations. On the other hand, Cai *et al*. [7, 8] proposed certain essentially nonoscillatory approximations and one-side filters for fitting discontinuous data. In particular, Gottlieb *et al*. [9] and Gottlieb and Shu [10–13] recovered the spectral accuracy by using Gegenbauer approximation. But so far, there is no work concerning spectral method for singular problems. In fact, the Gegenbauer approximation can also be applied to such problems. Guo [14] used it for some

0022-247X/00 $35.00

problems with symmetric singularities. However, in most practical problems the singularities are not symmetric, and so the Jacobi approximation is preferable. This paper is devoted to Jacobi approximations and their applications. The main idea is to fit singular solutions by Jacobi polynomials, to compare numerical solutions with some unusual orthogonal projections of exact solutions, and to measure the errors in certain Hilbert spaces. In the next section, several weighted inverse inequalities and Poincaré inequalities are given, and some approximation results are obtained. In particular, we consider the Jacobi approximations in certain specific Hilbert spaces, instead of Sobolev spaces. All results in that section play important roles in the numerical analysis of Jacobi spectral method for singular problems. The final part of this paper is for the applications of this new approach. We first consider a linear problem. Then we take the logistic equation in biology as an example, to show how to deal with nonlinear problems. The stabilities and the convergences of proposed schemes are proved strictly. The main idea and techniques used in this paper are also useful for other singular problems in multiple-dimensional spaces.

2. JACOBI APPROXIMATION

Let $\Lambda = \{x \mid |x| < 1\}$ and let $\chi(x)$ be a certain weight function in the usual sense. For $1 \le p \le \infty$, set

$$L^p_\chi(\Lambda) = \left\{ v \mid v \text{ is measurable and } \|v\|_{L^p_\chi} < \infty \right\},$$

where

$$\|v\|_{L^p_\chi} = \begin{cases} \left(\int_\Lambda |v(x)|^p \chi(x)\, dx \right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x \in \Lambda} |v(x)|, & p = \infty. \end{cases}$$

In particular, we denote by $(u, v)_\chi$ and $\|v\|_\chi$ the inner product and the norm of the Hilbert space $L^2_\chi(\Lambda)$. Further let $\partial_x v(x) = \frac{\partial}{\partial x} v(x)$, and for nonnegative integer m, define

$$H^m_\chi(\Lambda) = \left\{ v \mid \partial^k_x v \in L^2_\chi(\Lambda), 0 \le k \le m \right\},$$

equipped with the semi-norm $|v|_{m,\chi}$ and the norm $\|v\|_{m,\chi}$ as usual. For any real $r \ge 0$, we define the space $H^r_\chi(\Lambda)$ by the space interpolation as in Adams [15]. Let $\mathscr{D}(\Lambda)$ be the set of all infinitely differentiable functions with compact supports in Λ, and let $H^r_{0,\chi}(\Lambda)$ be its closure in $H^r_\chi(\Lambda)$. If $\chi(x) \equiv 1$, then we denote $H^r_\chi(\Lambda)$, $H^r_{0,\chi}(\Lambda)$, $|v|_{r,\chi}$, $\|v\|_{r,\chi}$, $\|v\|_\chi$, and $(u,v)_\chi$ by $H^r(\Lambda)$, $H^r_0(\Lambda)$, $|v|_r$, $\|v\|_r$, $\|v\|$, and (u,v), respectively. In addition, $\|v\|_\infty = \|v\|_{L^\infty(\Lambda)}$.

We now recall some properties of the Jacobi polynomials $J_l^{(\alpha,\beta)}(x)$, defined by

$$(1-x)^\alpha(1+x)^\beta J_l^{(\alpha,\beta)}(x) = \frac{(-1)^l}{2^l l!}\,\partial_x^l\big((1-x)^{l+\alpha}(1+x)^{l+\beta}\big).$$

They are the eigenfunctions of the singular Sturm–Liouville problem

$$\begin{aligned}\partial_x\big((1-x)^{\alpha+1}(1+x)^{\beta+1}\,\partial_x v(x)\big) &+ \lambda(1-x)^\alpha(1+x)^\beta v(x)\\ &= 0, \qquad x \in \Lambda. \end{aligned} \tag{2.1}$$

The corresponding eigenvalues are $\lambda_l^{(\alpha,\beta)} = l(l+\alpha+\beta+1)$. They fulfill the recurrence relations (see Askey [16] and Rainville [17])

$$\begin{aligned} 2(l+\alpha+1)J_l^{(\alpha,\beta)}(x) &- 2(l+1)J_{l+1}^{(\alpha,\beta)}(x)\\ &= (2l+\alpha+\beta+2)(1-x)J_l^{(\alpha+1,\beta)}(x),\\ J_l^{(\alpha,\beta-1)}(x) &- J_l^{(\alpha-1,\beta)}(x) = J_{l-1}^{(\alpha,\beta)}(x), \end{aligned}$$

and

$$(l+\alpha+\beta)J_l^{(\alpha,\beta)}(x) = (l+\beta)J_l^{(\alpha,\beta-1)}(x) + (l+\alpha)J_l^{(\alpha-1,\beta)}(x), \tag{2.2}$$

$$\partial_x J_l^{(\alpha,\beta)}(x) = \tfrac{1}{2}(l+\alpha+\beta+1)J_{l-1}^{(\alpha+1,\beta+1)}(x). \tag{2.3}$$

Let $\Gamma(x)$ be the Gamma function. We have that (see Askey [16])

$$\begin{aligned} J_l^{(\alpha,\beta)}(x) &= \frac{\Gamma(l+\beta+1)}{\Gamma(l+\alpha+\beta+1)}\sum_{k=0}^{l}\frac{(2k+\alpha+\beta)\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta+1)}\\ &\qquad \times J_k^{(\alpha-1,\beta)}(x), \end{aligned} \tag{2.4}$$

$$\begin{aligned} J_l^{(\alpha,\beta)}(x) &= \frac{\Gamma(l+\alpha+1)}{\Gamma(l+\alpha+\beta+1)}\sum_{k=0}^{l}(-1)^{l-k}\\ &\qquad \times \frac{(2k+\alpha+\beta)\Gamma(k+\alpha+\beta)}{\Gamma(k+\alpha+1)}J_k^{(\alpha,\beta-1)}(x). \end{aligned} \tag{2.5}$$

Also, note that

$$J_l^{(\alpha,\beta)}(-x) = (-1)^l J_l^{(\beta,\alpha)}(x), \qquad J_l^{(\alpha,\beta)}(1) = \frac{\Gamma(l+\alpha+1)}{l!\,\Gamma(\alpha+1)}.$$

Let

$$\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta.$$

For any real numbers $\alpha, \beta > -1$, the set $\{J_l^{(\alpha,\beta)}(x)\}$ is the $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal system,

$$\left(J_l^{(\alpha,\beta)}, J_m^{(\alpha,\beta)}\right)_{\chi^{(\alpha,\beta)}} = \gamma_l^{(\alpha,\beta)} \delta_{l,m}, \tag{2.6}$$

where $\delta_{l,m}$ is the Kronecker function, and

$$\gamma_l^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{(2l+\alpha+\beta+1)\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}.$$

In this paper, we suppose that $\alpha, \beta > -1$, except that a few formulas are valid for $\alpha = -1$ or $\beta = -1$. For any $v \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$,

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l^{(\alpha,\beta)} J_l^{(\alpha,\beta)}(x),$$

where $\hat{v}_l^{(\alpha,\beta)}$ is the Jacobi coefficient,

$$\hat{v}_l^{(\alpha,\beta)} = \frac{1}{\gamma_l^{(\alpha,\beta)}} \int_\Lambda v(x) J_l^{(\alpha,\beta)}(x) \chi^{(\alpha,\beta)}(x)\, dx. \tag{2.7}$$

Now let N be any positive integer, and let $\mathscr{P}_N$ be the set of all algebraic polynomials of degree at most N. ${}_0\mathscr{P}_N = \{v \mid v \in \mathscr{P}_N,\ v(-1) = 0\}$ and $\mathscr{P}_N^0 = \{v \mid v \in \mathscr{P}_N,\ v(-1) = v(1) = 0\}$. Denote by c a generic positive constant independent of any function and N.

In numerical analysis, we need some inverse inequalities. Let ϕ_l be an algebraic polynomial of degree l, and let the set of ϕ_l be an orthogonal system in $L^2_\chi(\Lambda)$. Let $\mathscr{L}$ be a linear operator defined on $\mathscr{P}_N$. $\mathscr{L}$ is said to be of (p,q) type, if there exists a positive constant d depending only on p, q, and N such that $\|\mathscr{L}\phi\|_{L^q_\chi} \le d\|\phi\|_{L^p_\chi}$ for any $\phi \in \mathscr{P}_N$. According to the Riesz–Thorin theorem, we know that if $\mathscr{L}$ is of both (p_1, q_1) type and (p_2, q_2) type for $1 \le p_1, p_2 < \infty$, $1 \le q_1, q_2 \le \infty$, then for

$$p = \frac{p_1 p_2}{(1-\theta)p_2 + \theta p_1}, \quad q = \frac{q_1 q_2}{(1-\theta)q_2 + \theta q_1}, \qquad 0 \le \theta \le 1,$$

the operator $\mathscr{L}$ is also of (p,q) type. If in addition $\|\mathscr{L}\phi\|_{L_\chi^{q_j}} \le d_j \|\phi\|_{L_\chi^{p_j}}$, $j = 1, 2$, then

$$\|\mathscr{L}\phi\|_{L_\chi^q} \le c(p_1, p_2) d_1^{1-\theta} d_2^{\theta} \|\phi\|_{L_\chi^p},$$

where $c(p_1, p_2)$ is a positive constant depending only on p_1 and p_2. By using the above fact, Guo [6] proved the following result.

LEMMA 2.1. *If for a certain positive constant c_0 and real number δ,*

$$\|\phi_0\|_\infty \le c_0, \quad \|\phi_l\|_\infty \le c_0 l^\delta \|\phi_l\|_\chi, \qquad l \ge 1,$$

then for any $\phi \in \mathscr{P}_N$ and all $1 \le p \le q \le \infty$,

$$\|\phi\|_{L_\chi^q} \le c\sigma^{1/p - 1/q}(N) \|\phi\|_{L_\chi^p},$$

where $\sigma(N) = N^{2\delta+1}$ for $\delta > -\frac{1}{2}$, $\sigma(N) = \ln N$ for $\delta = -\frac{1}{2}$, and $\sigma(N) = 1$ for $\delta < -\frac{1}{2}$.

THEOREM 2.1. *For any $\phi \in \mathscr{P}_N$ and $1 \le p \le q \le \infty$,*

$$\|\phi\|_{L^q_{\chi^{(\alpha,\beta)}}} \le cN^{\sigma(\alpha,\beta)(1/p - 1/q)} \|\phi\|_{L^p_{\chi^{(\alpha,\beta)}}},$$

where

$$\sigma(\alpha, \beta) = \begin{cases} 2\max(\alpha, \beta) + 2, & \text{if } \max(\alpha,\beta) \ge -\frac{1}{2}, \\ 1, & \text{otherwise}. \end{cases}$$

Proof. By the Stirling formula (see Courant and Hilbert [18]),

$$\Gamma(s+1) = \sqrt{2\pi s}\, s^s e^{-s}\left(1 + O(s^{-1/5})\right), \tag{2.8}$$

and so (2.6) implies that $\|J_l^{(\alpha,\beta)}\|_{\chi^{(\alpha,\beta)}} = O(l^{-1/2})$. On the other hand, by Abramowitz and Stegun [19], $\|J_l^{(\alpha,\beta)}\|_\infty \le cl^{\max(\alpha,\beta,-1/2)}$. Therefore

$$\left\| J_l^{(\alpha,\beta)} \right\|_\infty \le cl^{\max(\alpha+1/2, \beta+1/2, 0)} \left\| J_l^{(\alpha,\beta)} \right\|_{\chi^{(\alpha,\beta)}}.$$

Finally by taking $\phi_l(x) = J_l^{(\alpha,\beta)}(x)$, $\chi(x) = \chi^{(\alpha,\beta)}(x)$, $c_0 = c$, $\delta = \max(\alpha + \frac{1}{2}, \beta + \frac{1}{2}, 0)$ in Lemma 2.1, we obtain the desired result.

THEOREM 2.2. *For any $\phi \in \mathscr{P}_N$ and $r \ge 0$,*

$$\|\phi\|_{r,\chi^{(\alpha,\beta)}} \le cN^{2r} \|\phi\|_{\chi^{(\alpha,\beta)}}.$$

If in addition $\alpha, \beta > r - 1$, then

$$\|\phi\|_{r,\chi^{(\alpha,\beta)}} \le cN^{r} \|\phi\|_{\chi^{(\alpha-r,\beta-r)}}.$$

Proof. Let

$$\phi(x) = \sum_{l=0}^{N} \hat{\phi}_l^{(\alpha,\beta)} J_l^{(\alpha,\beta)}(x).$$

For simplicity, let

$$E_l = \frac{\Gamma(l+\beta+2)}{\Gamma(l+\alpha+\beta+2)}, \qquad F_k = \frac{(2k+\alpha+\beta+2)\Gamma(k+\alpha+\beta+2)}{\Gamma(k+\beta+2)},$$

$$G_k = \frac{(2k+\alpha+\beta+2)\Gamma(k+\alpha+1)}{\Gamma(k+\beta+2)},$$

$$H_j = \frac{(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\alpha+1)},$$

$$\psi_{j,l} = \sum_{k=j}^{l} (-1)^k G_k.$$

By virtue of (2.3)–(2.5) and the fact that $\Gamma(x+1) = x\Gamma(x)$, we obtain

$$\begin{aligned}
\partial_x \phi(x) &= \frac{1}{2} \sum_{l=0}^{N-1} (l+\alpha+\beta+2) \hat{\phi}_{l+1}^{(\alpha,\beta)}(x) J_l^{(\alpha+1,\beta+1)}(x) \\
&= \frac{1}{2} \sum_{l=0}^{N-1} E_l \hat{\phi}_{l+1}^{(\alpha,\beta)} \left(\sum_{k=0}^{l} F_k J_k^{(\alpha,\beta+1)}(x) \right) \\
&= \frac{1}{2} \sum_{l=0}^{N-1} E_l \hat{\phi}_{l+1}^{(\alpha,\beta)} \left(\sum_{k=0}^{l} (-1)^k G_k \left(\sum_{j=0}^{k} (-1)^j H_j J_j^{(\alpha,\beta)}(x) \right) \right) \\
&= \frac{1}{2} \sum_{l=0}^{N-1} E_l \hat{\phi}_{l+1}^{(\alpha,\beta)} \left(\sum_{j=0}^{l} (-1)^j \psi_{j,l} H_j J_j^{(\alpha,\beta)}(x) \right) \\
&= \frac{1}{2} \sum_{j=0}^{N-1} (-1)^j H_j J_j^{(\alpha,\beta)}(x) \left(\sum_{l=j}^{N-1} E_l \psi_{j,l} \hat{\phi}_{l+1} \right).
\end{aligned}$$

Let

$$A_k = \frac{(2k+\alpha+\beta+3)\Gamma(k+\alpha+1)}{\Gamma(k+\beta+3)}.$$

It can be checked that

$$G_{k+1} - G_k = (\alpha - \beta) A_k. \tag{2.9}$$

By (2.8),

$$|A_k| \le c(k+1)^{\alpha-\beta-1}, \qquad |G_k| \le c(k+1)^{\alpha-\beta},$$
$$|\psi_{j,l}| \le c(l+1)^{\alpha-\beta}, \qquad E_l \le c(l+1)^{-\alpha}, \qquad H_j \le c(j+1)^{\beta+1}. \tag{2.10}$$

The above estimates with (2.6) lead to

$$\begin{aligned}\|\partial_x \phi\|^2_{\chi^{(\alpha,\beta)}} &\le c \sum_{j=0}^{N-1} (j+1)^{2\beta+1} \left(\sum_{l=j}^{N-1} (l+1)^{1-2\beta} \right) \|\phi\|^2_{\chi^{(\alpha,\beta)}} \\ &\le cN^4 \|\phi\|^2_{\chi^{(\alpha,\beta)}}.\end{aligned}$$

By repeating the above procedure, we find that for any nonnegative integer m,

$$\|\partial_x^m \phi\|_{\chi^{(\alpha,\beta)}} \le cN^{2m} \|\phi\|_{\chi^{(\alpha,\beta)}}. \tag{2.11}$$

For any real $r \ge 0$, we derive the desired resul by using the interpolation of Hilbert spaces (see Bergh and Löfström [20]).

We now prove the second result. Let $\alpha, \beta > r - 1$ and

$$\phi(x) = \sum_{l=0}^{N} \hat{\phi}_l^{(\alpha-1,\beta-1)} J_l^{(\alpha-1,\beta-1)}(x).$$

By (2.3),

$$\partial_x \phi(x) = \frac{1}{2} \sum_{l=0}^{N} (l + \alpha + \beta) \hat{\phi}_{l+1}^{(\alpha-1,\beta-1)} J_l^{(\alpha,\beta)}(x).$$

Thus

$$\|\partial_x \phi(x)\|^2_{\chi^{(\alpha,\beta)}} \le \frac{1}{4} \sum_{l=0}^{N-1} (l+\alpha+\beta)^2 \gamma_l^{(\alpha,\beta)} \left(\hat{\phi}_{l+1}^{(\alpha-1,\beta-1)}\right)^2. \tag{2.12}$$

By (2.6) and (2.8),

$$\max_{0 \le l \le N} \frac{(l+\alpha+\beta)^2 \gamma_l^{(\alpha,\beta)}}{\gamma_l^{(\alpha-1,\beta-1)}} \le cN^2.$$

Hence (2.12) reads

$$\|\partial_x \phi\|_{\chi^{(\alpha,\beta)}} \le cN\|\phi\|_{\chi^{(\alpha-1,\beta-1)}}.$$

Finally we complete the proof by induction and space interpolation.

Remark 2.1. By the Markov theorem (see Timan [21]),

$$\|\partial_x \phi\|_\infty \le N \min\Big((1-x^2)^{-1/2}, N\Big)\|\phi\|_\infty.$$

So using (2.11) and space interpolation, we assert that for any $\phi \in \mathscr{P}_N$, nonnegative integer m, and $2 \le p \le \infty$,

$$\|\partial_x^m \phi\|_{L^p_{\chi^{(\alpha,\beta)}}} \le cN^{2m}\|\phi\|_{L^p_{\chi^{(\alpha,\beta)}}}.$$

We now consider various orthogonal projections. The $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal projection $P_{N,\alpha,\beta}$: $L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathscr{P}_N$ is such a mapping that for any $v \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$,

$$\big(P_{N,\alpha,\beta}v - v, \phi\big)_{\chi^{(\alpha,\beta)}} = 0, \qquad \forall \phi \in \mathscr{P}_N.$$

For technical reasons, we introduce another Hilbert space. For any nonnegative integer r,

$$H^r_{\chi^{(\alpha,\beta)},A}(\Lambda) = \big\{v \mid v \text{ is measurable and } \|v\|_{r,\chi^{(\alpha,\beta)},A} < \infty\big\},$$

where

$$\|v\|_{r,\chi^{(\alpha,\beta)},A} = \left(\sum_{k=0}^{[\frac{r-1}{2}]} \left\|(1-x^2)^{\frac{r}{2}-k}\,\partial_x^{r-k} v\right\|^2_{\chi^{(\alpha,\beta)}} + \|v\|^2_{[\frac{r}{2}],\chi^{(\alpha,\beta)}}\right)^{1/2},$$

For any real $r \ge 0$, the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ is defined by space interpolation. Let

$$Av(x) = -\big(\chi^{(\alpha,\beta)}(x)\big)^{-1}\,\partial_x\Big((1-x)^{\alpha+1}(1+x)^{\beta+1}\,\partial_x v(x)\Big).$$

By the induction,

$$A^m v(x) = \sum_{k=0}^{m-1} (1-x^2)^{m-k} p_k(x,\alpha,\beta)\,\partial_x^{2m-k} v(x) + \sum_{k=0}^{m} q_k(x,\alpha,\beta)\,\partial_x^k v(x),$$

where $p_k(x,\alpha,\beta)$ and $q_k(x,\alpha,\beta)$ are some polynomials. So A^m is a continuous mapping from $H^{2m}_{\chi^{(\alpha,\beta)},A}(\Lambda)$ to $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$.

Next, for any nonnegative integer μ,

$$H^r_{\chi^{(\alpha,\beta)},*,\mu}(\Lambda) = \left\{v \mid \partial_x^\mu v \in H^{r-\mu}_{\chi^{(\alpha,\beta)},A}(\Lambda)\right\},$$
$$H^r_{\chi^{(\alpha,\beta)},**,\mu}(\Lambda) = \left\{v \mid v \in H^r_{\chi^{(\alpha,\beta)},*,k}(\Lambda), 0 \le k \le \mu\right\}$$

with norms

$$\|v\|_{r,\chi^{(\alpha,\beta)},*,\mu} = \|\partial_x^\mu v\|_{r-\mu,\chi^{(\alpha,\beta)},A},$$
$$\|v\|_{r,\chi^{(\alpha,\beta)},**,\mu} = \left(\sum_{k=0}^{\mu} \|v\|^2_{r,\chi^{(\alpha,\beta)},*,k}\right)^{1/2}.$$

For any real $\mu \ge 0$, we define the spaces $H^r_{\chi^{(\alpha,\beta)}\,*,\mu}(\Lambda)$ and $H^r_{\chi^{(\alpha,\beta)}\,**,\mu}(\Lambda)$ by the space interpolations. In particular, $\|v\|_{r,\chi^{(\alpha,\beta)},*} = \|v\|_{r,\chi^{(\alpha,\beta)},*,1}$.

THEOREM 2.3. *For any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and $r \ge 0$,*

$$\|P_{N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}} \le cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},A}.$$

Proof. We first assume $r = 2m$. By virtue of (2.1), (2.6), and integration by parts,

$$\begin{aligned}\hat{v}_l^{(\alpha,\beta)} &= -\left(\gamma_l^{(\alpha,\beta)}\lambda_l^{(\alpha,\beta)}\right)^{-1}\int_\Lambda v(x)\,\partial_x\Big((1-x)^{\alpha+1}(1+x)^{\beta+1} \\ &\qquad\qquad \times \partial_x J_l^{(\alpha,\beta)}(x)\Big)\,dx \\ &= \left(\gamma_l^{(\alpha,\beta)}\lambda_l^{(\alpha,\beta)}\right)^{-1}\int_\Lambda Av(x)J_l^{(\alpha,\beta)}(x)\,\chi^{(\alpha,\beta)}(x)\,dx \\ &= \left(\gamma_l^{(\alpha,\beta)}\right)^{-1}\left(\lambda_l^{(\alpha,\beta)}\right)^{-m}\int_\Lambda A^m v(x)J_l^{(\alpha,\beta)}(x)\,\chi^{(\alpha,\beta)}(x)\,dx. \quad (2.13)\end{aligned}$$

Thus

$$\begin{aligned}\|P_{N,\alpha,\beta}v - v\|^2_{\chi^{(\alpha,\beta)}} &= \sum_{l=N+1}^{\infty}\gamma_l^{(\alpha,\beta)}\left(\hat{v}_l^{(\alpha,\beta)}\right)^2 \le cN^{-4m}\|A^m v\|_{\chi^{(\alpha,\beta)}} \\ &\le cN^{-2r}\|v\|^2_{r,\chi^{(\alpha,\beta)},A}. \quad (2.14)\end{aligned}$$

Next let $r = 2m + 1$. We have from (2.1), (2.13), and integration by parts that

$$\begin{aligned}\hat{v}_l^{(\alpha,\beta)} &= \left(\gamma_l^{(\alpha,\beta)}\right)^{-1}\left(\lambda_l^{(\alpha,\beta)}\right)^{-m-1} \\ &\quad \times \int_\Lambda \partial_x A^m v(x)\, \partial_x J_l^{(\alpha,\beta)}(x)\, \chi^{(\alpha+1,\beta+1)}(x)\, dx \\ &= \tfrac{1}{2}(l+\alpha+\beta+1)\left(\gamma_l^{(\alpha,\beta)}\right)^{-1}\left(\lambda_l^{(\alpha,\beta)}\right)^{-m-1} \\ &\quad \times \int_\Lambda \partial_x A^m v(x) J_{l-1}^{(\alpha+1,\beta+1)}(x)\, \chi^{(\alpha+1,\beta+1)}(x)\, dx.\end{aligned}$$

Thus by (2.6) and the Stirling formula,

$$\begin{aligned}&\|P_{N,\alpha,\beta}v - v\|^2_{\chi^{(\alpha,\beta)}} \\ &\quad \le c \max_{N+1\le l\le\infty}\left(l^2\left(\gamma_l^{(\alpha,\beta)}\right)^{-1}\gamma_{l-1}^{(\alpha+1,\beta+1)}\left(\lambda_l^{(\alpha,\beta)}\right)^{-2m-2}\right)\|\partial_x A^m v\|^2_{\chi^{(\alpha+1,\beta+1)}} \\ &\quad \le cN^{-4m-2}\|\partial_x A^m v\|^2_{\chi^{(\alpha+1,\beta+1)}}.\end{aligned}$$

Moreover

$$\|\partial_x A^m v\|_{\chi^{(\alpha+1,\beta+1)}} \le c\|v\|_{2m+1,\chi^{(\alpha,\beta)},A}.$$

Therefore (2.14) holds too.

Finally we complete the proof by using the space interpolation.

Note that Bernardi and Maday [22] considered the Jacobi approximation for $\alpha = \beta$. Theorem 2.3 with $\alpha = \beta$ improves Theorem 20.1 in Bernardi and Maday [22].

In general, $P_{N,\alpha,\beta}\,\partial_x v(x) \ne \partial_x P_{N,\alpha,\beta} v(x)$. But we have the following result.

LEMMA 2.2. *If $\alpha + r > 1$ or $\beta + r > 1$, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda) \cap L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$ and $r \ge 1$,*

$$\|P_{N,\alpha,\beta}\,\partial_x v - \partial_x P_{N,\alpha,\beta} v\|_{\chi^{(\alpha,\beta)}} \le cN^{2-r}\left(\|v\|_{r,\chi^{(\alpha,\beta)},*} + \|v\|_{\chi^{(\alpha,\beta)}}\right).$$

In particular, for any $\alpha = \beta > -1$,

$$\|P_{N,\alpha,\beta}\,\partial_x v - \partial_x P_{N,\alpha,\beta} v\|_{\chi^{(\alpha,\beta)}} \le cN^{3/2-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

Moreover, if $\alpha + r > 1$ *and* $\beta + r > 1$, *then for any* $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ *and* $r \geq 1$,

$$\|P_{N,\alpha,\beta}\,\partial_x v - \partial_x P_{N,\alpha,\beta} v\|_{\chi^{(\alpha,\beta)}} \leq cN^{2-r}\|v\|_{r,\chi^{(\alpha,\beta)},A}.$$

Proof. By using the same notation and the same argument as in the proof of Theorem 2.2, we find that

$$\partial_x P_{N,\alpha,\beta} v(x) = \frac{1}{2}\sum_{j=0}^{N-1}(-1)^j H_j J_j^{(\alpha,\beta)}(x)\left(\sum_{l=j}^{N-1} E_l \psi_{j,l}\hat{v}_{l+1}^{(\alpha,\beta)}\right).$$

Let

$$\partial_x v(x) = \sum_{j=0}^{\infty} a_j J_j^{(\alpha,\beta)}(x).$$

Then we derive in the same way that

$$a_j = \frac{1}{2}(-1)^j H_j \sum_{l=j}^{\infty} E_l \psi_{j,l}\hat{v}_{l+1}^{(\alpha,\beta)}.$$

Thus

$$P_{N,\alpha,\beta}\,\partial_x v(x) = \frac{1}{2}\sum_{j=0}^{N}(-1)^j H_j J_j^{(\alpha,\beta)}(x)\left(\sum_{l=j}^{\infty} E_l \psi_{j,l}\hat{v}_{l+1}^{(\alpha,\beta)}\right).$$

Consequently

$$\begin{aligned} &P_{N,\alpha,\beta}\,\partial_x v(x) - \partial_x P_{N,\alpha,\beta} v(x) \\ &\qquad = \frac{1}{2}\sum_{j=0}^{N}(-1)^j H_j J_j^{(\alpha,\beta)}(x)\left(\sum_{l=N}^{\infty} E_l \psi_{j,l}\hat{v}_{l+1}^{(\alpha,\beta)}\right). \end{aligned}$$

Since $\psi_{j,l} = \psi_{j,N} + \psi_{N,l}$, we obtain from (2.9) that

$$P_{N,\alpha,\beta}\,\partial_x v(x) - \partial_x P_{N,\alpha,\beta} v(x) = L_N(x) + M_N(x),$$

where

$$L_N(x) = a_N \phi_N(x), \qquad \phi_N(x) = H_N^{-1}\sum_{j=0}^{N}(-1)^{j+N} H_j J_j^{(\alpha,\beta)}(x),$$

$$M_N(x) = \frac{1}{2}\sum_{j=0}^{N}(-1)^j H_j J_j^{(\alpha,\beta)}(x)\left(\sum_{l=N}^{\infty} E_l \psi_{j,N}\hat{v}_{l+1}^{(\alpha,\beta)}\right).$$

By (2.6) and Theorem 2.3,

$$|a_N| \le c\left(\gamma_N^{(\alpha,\beta)}\right)^{-1/2} \|P_{N,\alpha,\beta}\, \partial_x v - \partial_x v\|_{\chi^{(\alpha,\beta)}}$$
$$\le cN^{3/2-r}\|\partial_x v\|_{r-1,\chi^{(\alpha,\beta)},A} \le cN^{3/2-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

Moreover (2.6) and (2.8) imply that $\|\phi_N\|_{\chi^{(\alpha,\beta)}} \le c$. Hence

$$\|L_N\|_{\chi^{(\alpha,\beta)}} \le cN^{3/2-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

Next, let

$$B_{j,l} = (-1)^{j-1} \sum_{k=0}^{[l-j-1]/2} \left(K_{2k+j+1} - K_{2k+j}\right) = (-1)^{j-1} \sum_{k=0}^{[l-j-1]/2} A_{2k+j}.$$

Furthermore by (2.9),

$$\psi_{j,N} = \begin{cases} (\alpha-\beta)B_{j,N}, & \text{if } N+j \text{ is odd,} \\ (\alpha-\beta)B_{j,N} + (-1)^N G_N, & \text{if } N+j \text{ is even.} \end{cases}$$

Thus by (2.10),

$$|\psi_{j,N}| \le cN^{\alpha-\beta}.$$

Without loss of generality, let r be any even integer. If $\alpha + r > 1$, then we obtain from (2.6), (2.10), and (2.13) that

$$\|M_N\|^2_{\chi^{(\alpha,\beta)}}$$
$$\le c\sum_{j=0}^{N} N^{2\alpha-2\beta} H_j^2 \gamma_j^{(\alpha,\beta)} \left(\sum_{l=N}^{\infty} l^{-2r} E_l^2 \left(\gamma_{l+1}^{(\alpha,\beta)}\right)^{-1}\right)\left(\sum_{l=N}^{\infty}\left(\gamma_{l+1}^{(\alpha,\beta)}\right)^{-1}\right.$$
$$\left.\times\left(\int_\Lambda A^m v(x) J_l^{(\alpha,\beta)}(x)\chi^{(\alpha,\beta)}(x)\,dx\right)^2\right)$$
$$\le c\sum_{j=0}^{N} N^{2\alpha+1}\left(\sum_{l=N}^{\infty} l^{-2r-2\alpha+1}\|v\|^2_{r,\chi^{(\alpha,\beta)},A}\right)$$
$$\le cN^{4-2r}\|v\|^2_{r,\chi^{(\alpha,\beta)},A}.$$

The above statements lead to

$$\|P_{N,\alpha,\beta}\,\partial_x v - \partial_x P_{N,\alpha,\beta} v\|_{\chi^{(\alpha,\beta)}} \le cN^{2-r}\left(\|v\|_{r,\chi^{(\alpha,\beta)},*} + \|v\|_{\chi^{(\alpha,\beta)}}\right).$$

The same result is valid for $\beta + r > 1$.

We next consider the case with $\alpha = \beta > -1$. We have

$$\psi_{j,l} = \begin{cases} 0, & \text{if } j + l \text{ is odd}, \\ (-1)^l G_l, & \text{if } j + l \text{ is even}. \end{cases}$$

If $N + j$ is even, then $\psi_{j,l} = \psi_{N,l}$, and so

$$\tfrac{1}{2}(-1)^j H_j \sum_{l=N}^{\infty} E_l \psi_{j,l} \hat{v}_{l+1}^{(\alpha,\beta)} = (-1)^{N+j} H_j H_N^{-1} a_N.$$

If $N + j$ is odd, then $\psi_{j,N} = 0$, and $\psi_{j,l} = \psi_{N+1,l}$. Thus

$$\tfrac{1}{2}(-1)^j H_j \sum_{l=N}^{\infty} E_l \psi_{j,l} \hat{v}_{l+1}^{(\alpha,\beta)} = (-1)^{N+j+1} H_j H_{N+1}^{-1} a_{N+1}.$$

Finally, by an argument as in the estimation for $\|L_N\|_{\chi^{(\alpha,\beta)}}$, we get the second result.

We now prove the last result. By (2.2)–(2.5),

$$\begin{aligned} \partial_x J_{l+1}^{(\alpha,\beta)}(x) &= \tfrac{1}{2}(l + \alpha + \beta + 2) J_l^{(\alpha+1,\beta+1)}(x) \\ &= \tfrac{1}{2}(l + \beta + 1) J_l^{(\alpha+1,\beta)}(x) + \tfrac{1}{2}(l + \alpha + 1) J_l^{(\alpha,\beta+1)}(x) \\ &= \sum_{j=0}^{l} \Big(C_{j,l}(\beta) + (-1)^{l-j} C_{j,l}(\alpha) \Big) J_j^{(\alpha,\beta)}(x), \end{aligned}$$

where

$$C_{j,l}(\theta) = \frac{(l + \theta + 1)(2j + \alpha + \beta + 1)\Gamma(l + \theta + 1)\Gamma(j + \alpha + \beta + 1)}{2\Gamma(j + \theta + 1)\Gamma(l + \alpha + \beta + 2)}, \quad \theta = \alpha, \beta.$$

Thus

$$\begin{aligned} \partial_x v(x) &= \sum_{l=0}^{\infty} \hat{v}_{l+1}^{(\alpha,\beta)} \, \partial_x J_{l+1}^{(\alpha,\beta)}(x) \\ &= \sum_{j=0}^{\infty} \sum_{l=j}^{\infty} \Big(C_{j,l}(\beta) + (-1)^{l-j} C_{j,l}(\alpha) \Big) \hat{v}_{l+1}^{(\alpha,\beta)} J_j^{(\alpha,\beta)}(x). \end{aligned}$$

Similarly

$$\partial_x P_{N,\alpha,\beta} v(x) = \sum_{j=0}^{N-1} \sum_{l=j}^{N-1} \left(C_{j,l}(\beta) + (-1)^{l-j} C_{j,l}(\alpha) \right) \hat{v}_{l+1}^{(\alpha,\beta)} J_j^{(\alpha,\beta)}(x).$$

Let

$$D_{1,N}(x) = \sum_{j=0}^{N} \sum_{l=N}^{\infty} C_{j,l}(\beta) \hat{v}_{l+1}^{(\alpha,\beta)} J_j^{(\alpha,\beta)}(x),$$

$$D_{2,N}(x) = \sum_{j=0}^{N} \sum_{l=N}^{\infty} (-1)^{l-j} C_{j,l}(\alpha) \hat{v}_{l+1}^{(\alpha,\beta)} J_j^{(\alpha,\beta)}(x).$$

Then

$$P_{N,\alpha,\beta}\, \partial_x v(x) - \partial_x P_{N,\alpha,\beta} v(x) = D_{1,N}(x) + D_{2,N}(x).$$

Moreover (2.10) implies that $|C_{j,l}(\theta)| \le c l^{-\theta}(j+1)^{\theta+1}$, $\theta = \alpha, \beta$. If $\beta + r > 1$, then we obtain from (2.6) and (2.13) that

$$\begin{aligned} \|D_{1,N}\|^2_{\chi^{(\alpha,\beta)}} &\le c \sum_{j=0}^{N} (j+1)^{2\beta+2} \gamma_j^{(\alpha,\beta)} \sum_{l=N}^{\infty} l^{-2r-2\beta+1} \|v\|^2_{r,\chi^{(\alpha,\beta)},A} \\ &\le c N^{4-2r} \|v\|^2_{r,\chi^{(\alpha,\beta)},A}. \end{aligned}$$

If $\alpha + r > 1$, then the same estimate is valid for $\|D_{2,N}\|^2_{\chi^{(\alpha,\beta)}}$. The proof is complete.

THEOREM 2.4. If $\alpha + r > 1$ or $\beta + r > 1$, then for any $v \in H^r_{\chi^{(\alpha,\beta)},**,\mu}(\Lambda)$, $r \ge 1$, and $\mu \le r$,

$$\|P_{N,\alpha,\beta} v - v\|_{\mu,\chi^{(\alpha,\beta)}} \le c N^{\sigma(\mu,r)} \|v\|_{r,\chi^{(\alpha,\beta)},**,\mu},$$

where

$$\sigma(\mu, r) = \begin{cases} 2\mu - r, & \text{for } \mu \ge 0, \\ \mu - r, & \text{for } \mu < 0. \end{cases}$$

In particular, for any $\alpha = \beta > -1$, *the above result is valid with*

$$\sigma(\mu, r) = \begin{cases} 2\mu - r - \frac{1}{2}, & \text{for } \mu > 1, \\ \frac{3}{2}\mu - r, & \text{for } 0 \le \mu \le 1, \\ \mu - r, & \text{for } \mu < 0. \end{cases}$$

Proof. Theorem 2.3 gives the desired result for $\mu = 0$. Now let $\mu > 0$. Space interpolation allows us to consider positive integer μ only. We now use induction. Assume that the conclusion is true for $\mu - 1$. Then by Lemma 2.2 and Theorem 2.2,

$$
\begin{aligned}
&\|P_{N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)}} \\
&\quad \le \|P_{N,\alpha,\beta}\,\partial_x v - \partial_x v\|_{\mu-1,\chi^{(\alpha,\beta)}} + \|P_{N,\alpha,\beta}\,\partial_x v - \partial_x P_{N,\alpha,\beta}v\|_{\mu-1,\chi^{(\alpha,\beta)}} \\
&\qquad + \|P_{N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}} \\
&\quad \le cN^{\sigma(\mu-1,r-1)}\sum_{k=0}^{\mu-1}\|\partial_x v\|_{r-1,\chi^{(\alpha,\beta)},*,k} \\
&\qquad + cN^{2\mu-2}\|P_{N,\alpha,\beta}\,\partial_x v - \partial_{xP_{N,\alpha,\beta}}v\|_{\chi^{(\alpha,\beta)}} + cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},A} \\
&\quad \le cN^{\sigma(\mu-1,r-1)}\sum_{k=1}^{\mu}\|\partial_x v\|_{r-1,\chi^{(\alpha,\beta)},*,k-1} \\
&\qquad + cN^{\sigma(\mu,r)}\big(\|v\|_{r,\chi^{(\alpha,\beta)},*} + \|v\|_{\chi^{(\alpha,\beta)}}\big) + cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},A}.
\end{aligned}
$$

Since $\sigma(\mu - 1, r - 1) \le \sigma(\mu, r)$, and $\|\partial_x v\|_{r-1,\chi^{(\alpha,\beta)},*,k-1} = \|v\|_{r,\chi^{(\alpha,\beta)},*,k}$, the conclusion for $\mu > 0$ follows immediately. Finally a duality argument leads to the result for $\mu < 0$. So we obtain the first result.

We can prove the second result in the same manner.

We can define the norm $\|v\|_{r,\chi^{(\alpha,\beta)},A}$ in another way and derive another kind of approximation results. Indeed, by (2.1) and the definition of A, $AJ_l^{(\alpha,\beta)}(x) = \lambda_l^{(\alpha,\beta)}J_l^{(\alpha,\beta)}(x)$. For any nonnegative integer r, we define the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ with the semi-norm

$$|v|_{r,\chi^{(\alpha,\beta)},A} = (v, A^r v)^{1/2}_{\chi^{(\alpha,\beta)}} = \left(\sum_{l=1}^{\infty}\left(\lambda_l^{(\alpha,\beta)}\right)^r|\hat{v}_l^{(\alpha,\beta)}|^2\gamma_l^{(\alpha,\beta)}\right)^{1/2}$$

and the norm

$$\|v\|_{r,\chi^{(\alpha,\beta)},A} = \left(\sum_{l=0}^{\infty}\left(1 + \left(\lambda_l^{(\alpha,\beta)}\right)^r\right)|\hat{v}_l^{(\alpha,\beta)}|^2\gamma_l^{(\alpha,\beta)}\right)^{1/2}.$$

Clearly

$$\|v\|_{r,\chi^{(\alpha,\beta)},A} \sim \left(\sum_{k=0}^{r}|v|^2_{k,\chi^{(\alpha,\beta)},A}\right)^{1/2}.$$

For any real $r > 0$, we define the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and its norm by space interpolation. Then it is not difficult to show that

$$\|P_{N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)},A} \le cN^{\mu-r}|v|_{r,\chi^{(\alpha,\beta)},A}.$$

By replacing v by $v - \phi$ in the above formula, we deduce that for any $\phi \in \mathscr{P}_N$,

$$\|P_{N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)},A} \le cN^{\mu-r}|v - \phi|_{r,\chi^{(\alpha,\beta)},A}$$

and so for $N \gg r$,

$$\|P_{N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)},A} \le cN^{\mu-r} \inf_{\phi\in\mathscr{P}_N} |v - \phi|_{r,\chi^{(\alpha,\beta)},A}.$$

By this fact, we can derive a more precise estimate. To this end, let $J^{(\alpha,\beta)}_{l,k}(x) = \partial^k_x J^{(\alpha,\beta)}_l(x)$, and then formally

$$\partial^k_x v(x) = \sum_{l=0}^{\infty} \hat{v}^{(\alpha,\beta)}_l J^{(\alpha,\beta)}_{l,k}(x).$$

By virtue of (2.3),

$$J^{(\alpha,\beta)}_{l,k}(x) = \frac{\Gamma(l+\alpha+\beta+k+1)}{2^k\Gamma(l+\alpha+\beta+1)} J^{(\alpha+k,\beta+k)}_{l-k}(x).$$

Hence $J^{(\alpha,\beta)}_{l,k}(x)$ is the same as $J^{(\alpha+k,\beta+k)}_{l-k}(x)$, apart from a constant. Moreover

$$\lambda^{(\alpha+k,\beta+k)}_{l-k} = \lambda^{(\alpha,\beta)}_l - \lambda^{(\alpha,\beta)}_k.$$

Therefore (2.1) implies that

$$\partial_x\left(\chi^{(\alpha+k+1,\beta+k+1)}(x)\,\partial_x J^{(\alpha,\beta)}_{l,k}(x)\right) + \left(\lambda^{(\alpha,\beta)}_l - \lambda^{(\alpha,\beta)}_k\right)\chi^{(\alpha+k,\beta+k)}(x) J^{(\alpha,\beta)}_{l,k}(x) = 0.$$

Multiplying the above equality by $J^{(\alpha,\beta)}_{l,k}(x)$ and integrating the result by parts, we find that

$$\|J^{(\alpha,\beta)}_{l,k}\|^2_{\chi^{(\alpha+k,\beta+k)}} = \cdots = c^{(\alpha,\beta)}_{l,k}\gamma^{(\alpha,\beta)}_l,$$

where

$$c^{(\alpha,\beta)}_{l,k} = \prod_{j=0}^{k-1}\left(\lambda^{(\alpha,\beta)}_l - \lambda^{(\alpha,\beta)}_j\right).$$

Let

$$d_{l,k}^{(\alpha,\beta)} = \frac{c_{l,k}^{(\alpha,\beta)}}{\left(\lambda_l^{(\alpha,\beta)}\right)^k}.$$

Clearly for $l \gg k$, $d_{l,k}^{(\alpha,\beta)} \geq c_k > 0$. Finally we obtain

$$\begin{aligned}
&\|P_{N,\alpha,\beta}v - v\|^2_{\mu,\chi^{(\alpha,\beta)},A} \\
&\quad \leq cN^{2\mu-2r}|v - P_{N,\alpha,\beta}v|^2_{r,\chi^{(\alpha,\beta)},A} \\
&\quad = cN^{2\mu-2r}\sum_{l=N+1}^{\infty}\left(\lambda_l^{(\alpha,\beta)}\right)^r|\hat{v}_l^{(\alpha,\beta)}|^2\gamma_l^{(\alpha,\beta)} \\
&\quad = cN^{2\mu-2r}\sum_{l=N+1}^{\infty}\frac{\left(\lambda_l^{(\alpha,\beta)}\right)^r}{c_{l,r}}|\hat{v}_l^{(\alpha,\beta)}|^2\|J_{l,r}^{(\alpha,\beta)}\|^2_{\chi^{(\alpha+r,\beta+r)}} \\
&\quad \leq \frac{c}{c_r}N^{2\mu-2r}\sum_{l=N+1}^{\infty}|\hat{v}_l^{(\alpha,\beta)}|^2\|J_{l,r}^{(\alpha,\beta)}\|^2_{\chi^{(\alpha+r,\beta+r)}} \\
&\quad = \frac{c}{c_r}N^{2\mu-2r}\|\partial_x^r v\|^2_{\chi^{(\alpha+r,\beta+r)}}.
\end{aligned}$$

In the following, we shall use the first kind of definition for the norm of the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$, since it is more natural in its applications to singular differential equations.

As is well known, we usually consider the $H^r_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal projection in numreical analysis of differential equations. But in many practical problems, the coefficients of derivatives of different orders may degenerate in different ways. Also by certain suitable variable transformations, differential equations in unbounded domains might be changed to be some singular problems in bounded domains. In these cases, it is not possible to compare the approximate solutions with the exact solutions in Sobolev spaces. Whereas it might be carried out in certain Hilbert spaces. To do this, let $\alpha, \beta, \gamma, \delta > -1$, and introduce the space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$, $0 \leq \mu \leq 1$. For $\mu = 0$, $H^0_{\alpha,\beta,\gamma,\delta}(\Lambda) = L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$. For $\mu = 1$,

$$H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \{v \mid v \text{ is measurable and } \|v\|_{1,\alpha,\beta,\gamma,\delta} < \infty\},$$

where

$$\|v\|_{1,\alpha,\beta,\gamma,\delta} = \left(|v|^2_{1,\chi^{(\alpha,\beta)}} + \|v\|^2_{\chi^{(\gamma,\delta)}}\right)^{1/2}.$$

For $0 < \mu < 1$, the space $H^{\mu}_{\alpha,\beta,\gamma,\delta}(\Lambda)$ is defined by space interpolation. Its norm is denoted by $\|v\|_{\mu,\alpha,\beta,\gamma,\delta}$. Let

$$a_{\alpha,\beta,\gamma,\delta}(u,v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (u,v)_{\chi^{(\gamma,\delta)}}, \qquad \forall u,v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda).$$

In particular, $a_{\alpha,\beta}(u,v) = a_{\alpha,\beta,\alpha,\beta}(u,v)$. The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta}$: $H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathscr{P}_N$ is such a mapping that for any $v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$,

$$a_{\alpha,\beta,\gamma,\delta}\left(P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi\right) = 0, \qquad \forall \phi \in \mathscr{P}_N.$$

In particular, $P^1_{N,\alpha,\beta} = P^1_{N,\alpha,\beta,\alpha,\beta}$.

For estimating the difference between $P^1_{N,\alpha,\beta,\gamma,\delta}v$ and v, we need the following lemma.

LEMMA 2.3. *For any $v \in H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$ with $v(0) = 0$, we have $\|v\|_{\chi^{(\gamma,\delta)}} \leq c|v|_{1,\chi^{(\alpha,\beta)}}$, provided*

$$\alpha \leq \gamma + 2, \qquad \beta \leq \delta + 2, \tag{2.15}$$

Proof. For any $x \in [0,1]$,

$$v^2(x)(1-x)^{\gamma+1} = \int_0^x \partial_y\left(v^2(y)(1-y)^{\gamma+1}\right) dy,$$

whence

$$\begin{aligned} v^2(x)(1-x)^{\gamma+1} &+ (\gamma+1)\int_0^x v^2(y)(1-y)^{\gamma}\, dy \\ &= 2\int_0^x v(y)\, \partial_y v(y)(1-y)^{\gamma+1}\, dy \\ &\leq 2\left(\int_0^x v^2(y)(1-y)^{\gamma}\, dy\right)^{1/2}\left(\int_0^x \left(\partial_y v(y)\right)^2(1-y)^{\gamma+2}\, dy\right)^{1/2}. \end{aligned} \tag{2.16}$$

Put

$$I_1(\gamma) = \int_0^1 v^2(x)(1-x)^{\gamma}\, dx,$$

$$I_2(\gamma) = \frac{4}{(\gamma+1)^2}\int_0^1 \left(\partial_x v(x)\right)^2(1-x)^{\gamma+2}\, dx,$$

$$I_3(\gamma,\delta) = \int_0^1 v^2(x)\chi^{(\gamma,\delta)}(x)\, dx,$$

$$I_4(\gamma,\delta) = \frac{4}{(\gamma+1)^2}\int_0^1 \left(\partial_x v(x)\right)^2 \chi^{(\gamma+2,\delta+2)}(x)\, dx.$$

Letting $x \to 1$ in (2.16), it follows that $I_1(\gamma) \le I_2(\gamma)$. Moreover

$$I_3(\gamma,\beta) \le \begin{cases} 2^\delta I_1(\gamma), & \text{for } \delta \ge 0, \\ I_1(\gamma), & \text{for } \delta < 0, \end{cases}$$

$$I_2(\gamma) \le I_4(\gamma,\delta).$$

Similar estimates are valid on the subinterval $[-1,0]$. The previous statements imply that $\|v\|_{\chi^{(\gamma,\delta)}} \le c|v|_{1,\chi^{(\gamma+2,\delta+2)}}$. Therefore $\|v\|_{\chi^{(\gamma,\delta)}} \le c|v|_{1,\chi^{(\alpha,\beta)}}$.

THEOREM 2.5. *If* (2.15) *holds, then for any* $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ *with* $r \ge 1$,

$$\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{1,\alpha,\beta,\gamma,\delta} \le cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

If, in addition,

$$\alpha \le \gamma + 1, \quad \beta \le \delta + 1, \tag{2.17}$$

then for all $0 \le \mu \le 1$,

$$\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{\mu,\alpha,\beta,\gamma,\delta} \le cN^{\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

Proof. Let

$$\phi(x) = \int_{-1}^{x} P_{N-1,\alpha,\beta}\,\partial_x v(y)\,dy + \xi, \tag{2.18}$$

where ξ is chosen in such a way that $v(0) = \phi(0)$. By the projection theorem, Lemma 2.3 and Theorem 2.3,

$$\begin{aligned}\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{1,\alpha,\beta,\gamma,\delta} &\le \|\phi - v\|_{1,\alpha,\beta,\gamma,\delta} \le c|\phi - v|_{1,\chi^{(\alpha,\beta)}} \\ &\le c\|P_{N-1,\alpha,\beta}\,\partial_x v - \partial_x v\|_{\chi^{(\alpha,\beta)}} \\ &\le cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.\end{aligned} \tag{2.19}$$

Now let (2.17) hold. Let $g \in L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$ and consider the auxiliary problem

$$a_{\alpha,\beta,\gamma,\delta}(w,z) = (g,z)_{\chi^{(\gamma,\delta)}}, \qquad \forall z \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda). \tag{2.20}$$

Taking $z = w$ in (2.20), we get that $\|w\|_{1,\alpha,\beta,\gamma,\delta} \le c\|g\|_{\chi^{(\gamma,\delta)}}$. Now let $w(x)$ vary in $\mathscr{D}(\Lambda)$, and so in the sense of distributions,

$$-\partial_x\big(\partial_x w(x)\,\chi^{(\alpha,\beta)}(x)\big) = (g(x) - w(x))\,\chi^{(\gamma,\delta)}(x). \tag{2.21}$$

If $\alpha, \beta > 0$, then $\partial_x w(x)\chi^{(\alpha,\beta)}(x) \to 0$ as $|x| \to 1$. If $-1 < \alpha, \beta \le 0$, then integrating (2.21) yields

$$\begin{aligned}&\left|\partial_x w(x_2)\chi^{(\alpha,\beta)}(x_2) - \partial_x w(x_1)\chi^{(\alpha,\beta)}(x_1)\right| \\ &\qquad \le \|g - w\|_{\chi^{(\gamma,\delta)}} \left(\int_{x_1}^{x_2} \chi^{(\gamma,\delta)}(x)\,dx\right)^{1/2}.\end{aligned}$$

Thus $\partial_x w(x)\chi^{(\alpha,\beta)}(x)$ is meaningful at $x = \pm 1$. Multiplying (2.21) by any $z \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$ and integrating the resulting equality by parts, we have from (2.20) that

$$\begin{aligned}&\partial_x w(1) z(1)\chi^{(\alpha,\beta)}(1) - \partial_x w(-1) z(-1)\chi^{(\alpha,\beta)}(-1) \\ &\quad = \int_\Lambda \left(\partial_x w(x)\, \partial_x z(x)\chi^{(\alpha,\beta)}(x) - (g(x) - w(x)) z(x)\chi^{(\gamma,\delta)}(x)\right) dx \\ &\quad = 0, \qquad \forall z \in H^1_{\chi^{(\alpha,\beta)}}(\Lambda).\end{aligned}$$

Hence $\partial_x w(1)\chi^{(\alpha,\beta)}(1) = \partial_x w(-1)\chi^{(\alpha,\beta)}(-1) = 0$. Moreover by (2.21), we obtain

$$\begin{aligned}-\partial_x^2 w(x) = &-((\alpha + \beta)x + (\alpha - \beta))(1 - x^2)^{-1}\, \partial_x w(x) \\ &+ (g(x) - w(x))\chi^{(\gamma-\alpha,\delta-\beta)}(x).\end{aligned} \tag{2.22}$$

Let $\Lambda_1 = [0,1]$ and $\Lambda_2 = [-1,0]$. It can be verified that

$$\left\|\partial_x^2 w(1 - x^2)^{1/2}\right\|^2_{\chi^{(\alpha,\beta)}} \le D_1 + D_2, \tag{2.23}$$

where $D_1 = D_1(\Lambda_1) + D_1(\Lambda_2)$,

$$D_1(\Lambda_j) = 8(\alpha^2 + \beta^2)\int_{\Lambda_j} (\partial_x w(x))^2 \chi^{(\alpha-1,\beta-1)}(x)\,dx, \qquad j = 1,2,$$

and

$$D_2 = 2\left|\int_\Lambda (g(x) - w(x))^2 \chi^{(2\gamma-\alpha+1,2\delta-\beta+1)}(x)\,dx\right|.$$

Thanks to (2.17), $D_2 \le c\|g-w\|_{\chi^{(\gamma,\delta)}}$. So it remains to estimate D_2. By (2.17) and (2.21),

$$\begin{aligned}
D_1(\Lambda_1) &= 8(\alpha^2+\beta^2)\int_0^1 (1-x)^{-\alpha-1}(1+x)^{-\beta-1} \\
&\qquad\times\left(\int_x^1 (g(y)-w(y))\,\chi^{(\gamma,\delta)}(y)\,dy\right)^2 dx \\
&\le c\int_0^1 (1-x)^{-\alpha-1}\left(\int_x^1 (g(y)-w(y))\,\chi^{(\gamma,\delta)}(y)\,dy\right)^2 dx \\
&\le c\int_0^1 (1-x)^{1-\alpha}\left(\frac{1}{1-x}\int_x^1 (g(y)-w(y))\,\chi^{(\gamma,\delta)}(y)\,dy\right)^2 dx \\
&\le c\int_0^1 (1-x)^{-\gamma}\left(\frac{1}{1-x}\int_x^1 (g(y)-w(y))\,\chi^{(\gamma,\delta)}(y)\,dy\right)^2 dx.
\end{aligned}$$

By the Hardy inequality (see Hardy *et al.* [23]), for any measurable function $\phi(x)$, real numbers $a \le b$ and $d < 1$,

$$\int_a^b \left(\frac{1}{b-x}\int_x^b \phi(y)\,dy\right)^2 (b-x)^d\,dx \le \frac{4}{1-d}\int_a^b \phi^2(x)(b-x)^d\,dx. \tag{2.24}$$

Let $d=-\gamma$ and $\phi(x)=(g(x)-w(x))\chi^{(\gamma,\delta)}(x)$. We find from (2.17) and (2.24) that

$$D_1(\Lambda_1) \le c\int_0^1 (g(x)-w(x))^2\chi^{(\gamma,\delta)}(x)\,dx.$$

A similar estimate is valid on Λ_2. Therefore

$$\|\partial_x^2 w(1-x)^{1/2}\|_{\chi^{(\alpha,\beta)}} \le c\left(\|w\|_{\chi^{(\gamma,\delta)}}+\|g\|_{\chi^{(\gamma,\delta)}}\right) \le c\|g\|_{\chi^{(\gamma,\delta)}},$$

from which and (2.19), we get

$$\begin{aligned}
\|P^1_{N,\alpha,\beta,\gamma,\delta}w-w\|_{1,\alpha,\beta,\gamma,\delta} &\le cN^{-1}\|w\|_{2,\chi^{(\alpha,\beta)},*} \\
&\le cN^{-1}\|\partial_x w\|_{1,\chi^{(\alpha,\beta)},A} \le cN^{-1}\|g\|_{\chi^{(\gamma,\delta)}}.
\end{aligned}$$

Taking $z = P^1_{N,\alpha,\beta,\gamma,\delta} v - v$ in (2.20), we obtain

$$\begin{aligned}
&\left|\left(P^1_{N,\alpha,\beta,\gamma,\delta} v - v, g\right)_{\chi^{(\gamma,\delta)}}\right| \\
&\quad = \left| a_{\alpha,\beta,\gamma,\delta}\left(P^1_{N,\alpha,\beta,\gamma,\delta} v - v, w\right)\right| \\
&\quad = \left| a_{\alpha,\beta,\gamma,\delta}\left(P^1_{N,\alpha,\beta,\gamma,\delta} v - v, P^1_{N,\alpha,\beta,\gamma,\delta} w - w\right)\right| \\
&\quad \le \|P^1_{N,\alpha,\beta,\gamma,\delta} v - v\|_{1,\alpha,\beta,\gamma,\delta} \|P^1_{N,\alpha,\beta,\gamma,\delta} w - w\|_{1,\alpha,\beta,\gamma,\delta} \\
&\quad \le c N^{-r} \|g\|_{\chi^{(\gamma,\delta)}} \|v\|_{r,\chi^{(\alpha,\beta)},*}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\|P^1_{N,\alpha,\beta,\gamma,\delta} v - v\|_{\chi^{(\gamma,\delta)}} \\
&\quad = \sup_{\substack{g \in L^2_{\chi^{(\gamma,\delta)}} \\ g \neq 0}} \frac{\left(P^1_{N,\alpha,\beta,\gamma,\delta} v - v, g\right)_{\chi^{(\gamma,\delta)}}}{\|g\|_{\chi^{(\gamma,\delta)}}} \le c N^{-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}.
\end{aligned}$$

Finally the result for $0 < \mu < 1$ follows from space interpolation.

Remark 2.2. The special cases with $\alpha = \gamma = 1, 2$ and $\beta = \delta = 0$ were discussed in Guo [24, 25]. The other case, with $\alpha = \beta$, $\gamma = \delta = 0$, was considered in Guo [14].

In some practical problems arising in fluid dynamics, biology, and other fields, the unknown functions vanish at one of the extreme points, say $x = -1$. So we need other orthognoal projections. Let

$$ {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \left\{ v \mid v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } v(-1) = 0 \right\}.$$

The othogonal projection ${}_0P^1_N \colon {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to {}_0\mathscr{P}_N$ is such a mapping that for any $v \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$,

$$a_{\alpha,\beta,\gamma,\delta}\left({}_0P^1_{N,\alpha,\beta,\gamma,\delta} v - v, \phi\right) = 0 \qquad \forall \phi \in {}_0\mathscr{P}_N.$$

LEMMA 2.4. For any $v \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$, we have $\|v\|_{\chi^{(\gamma,\delta)}} \le c|v|_{1,\chi^{(\alpha,\beta)}}$, provided

$$\alpha \le \gamma + 2, \qquad \beta \le 0, \qquad \delta \ge 0. \tag{2.25}$$

Proof. As in the proof of Lemma 2.3,

$$\begin{aligned}
&v^2(x)(1-x)^{\gamma+1} + (\gamma+1)\int_{-1}^{x} v^2(y)(1-y)^{\gamma}\,dy \\
&\qquad = 2\int_{-1}^{x} v(y)\,\partial_y v(y)(1-y)^{\gamma+1}\,dy.
\end{aligned}$$

Letting $x \to 1$, it follows that

$$(\gamma+1)\int_\Lambda v^2(x)(1-x)^\gamma\,dx \le 2\left(\int_\Lambda v^2(x)(1-x)^{2\gamma-\alpha+2}\,dx\right)^{1/2}\left(\int_\Lambda (\partial_x v(x))^2(1-x)^\alpha\,dx\right)^{1/2}.$$

On the other hand,

$$\int_\Lambda v^2(x)\chi^{(\gamma,\delta)}(x)\,dx \le c\int_\Lambda v^2(x)(1-x)^\gamma\,dx,$$

$$\int_\Lambda (\partial_x v(x))^2(1-x)^\alpha\,dx \le c\int_\Lambda (\partial_x v(x))^2\chi^{(\alpha,\beta)}(x)\,dx.$$

The above statements lead to the conclusion.

Remark 2.3. For any $v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$ with $v(1)=0$, we have $\|v\|_{\chi^{(\gamma,\delta)}} \le c|v|_{1,\chi^{(\alpha,\beta)}}$, provided

$$\alpha \le 0, \qquad \beta \le \delta+2, \qquad \gamma \ge 0. \tag{2.26}$$

THEOREM 2.6. *If* (2.25) *holds, then for any* $v \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ *with* $r \ge 1$,

$$\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{1,\alpha,\beta,\gamma,\delta} \le cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

If, in addition, (2.17) *holds, then for all* $0 \le \mu \le 1$,

$$\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{\mu,\alpha,\beta,\gamma,\delta} \le cN^{\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

Proof. Let

$$\phi(x) = \int_{-1}^x P_{N-1,\alpha,\beta}\,\partial_y v(y)\,dy.$$

By the projection theorem, Lemma 2.4, and Theorem 2.3,

$$\begin{aligned}\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{1,\alpha,\beta,\gamma,\delta} &\le \|\phi - v\|_{1,\alpha,\beta,\gamma,\delta} \le c|\phi - v|_{1,\chi^{(\alpha,\beta)}}\\ &\le c\|P_{N-1,\alpha,\beta}\,\partial_x v - \partial_x v\|_{\chi^{(\alpha,\beta)}}\\ &\le cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.\end{aligned} \tag{2.27}$$

Now assume that (2.17) holds. Let $g \in L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$ and consider the auxiliary problem

$$a_{\alpha,\beta,\gamma,\delta}(w,z) = (g,z)_{\chi^{(\gamma,\delta)}}, \qquad \forall z \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda). \tag{2.28}$$

Taking $z = w$ in (2.28), we get $\|w\|_{1,\alpha,\beta,\gamma,\delta} \leq c\|g\|_{\chi^{(\gamma,\delta)}}$. Also in the sense of distributions, (2.21) holds and $\partial_x w(1)\chi^{(\alpha,\beta)}(1) = 0$. Moreover (2.22) and (2.23) hold. Finally by (2.24), (2.27), and an argument as in the last part of the proof of Theorem 2.5, we reach the second result.

Remark 2.4. If (2.26) holds, then for any $v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ with $v(1) = 0$ and $r \geq 1$, the corresponding result holds. But ${}_0P^1_{N,\alpha,\beta,\gamma,\delta}$ is now replaced by the orthogonal projection ${}^0P^1_{N,\alpha,\beta,\gamma,\delta}$, where ${}^0\mathscr{P}_N = \{v \mid v \in \mathscr{P}_N \text{ and } v(1) = 0\}$ and

$$a_{\alpha,\beta,\gamma,\delta}\left({}^0P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi\right) = 0, \qquad \forall \phi \in {}^0\mathscr{P}_N.$$

If, in addition, (2.17) holds, then the corresponding improved result holds.

When we study the movements of fluid flows in bounded domains with fixed nonslip walls, of the populations of budworms in bounded forests with lethal boundary conditions, and of some other topics, we meet homogeneous boundary conditions. In those caes, we have to consider another projection. Let

$$H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) = \left\{v \mid v \in H_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } v(-1) = v(1) = 0\right\}.$$

The orthogonal projection $P^{1,0}_{N,\alpha,\beta,\gamma,\delta}\colon H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathscr{P}^0_N$ is such a mapping that for any $v \in H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda)$,

$$a_{\alpha,\beta,\gamma,\delta}\left(P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v - v, \phi\right) = 0, \qquad \phi \in \mathscr{P}^0_N.$$

THEOREM 2.7. *If $\gamma \leq \alpha \leq \gamma + 1$, $\delta \leq \beta \leq \delta + 1$ and $\gamma, \delta < 1$, then for any $v \in H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ with $r \geq 2$,*

$$\|P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v - v\|_{1,\alpha,\beta,\gamma,\delta} \leq cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*,2}.$$

If, in addition, $\alpha = \gamma$, $\beta = \delta$ and $\alpha, \beta > 0$, then for all $0 \leq \mu \leq 1$.

$$\|P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v - v\|_{\mu,\alpha,\beta,\gamma,\delta} \leq cN^{\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*,2}.$$

Proof. Let

$$\phi^*(x) = \int_{-1}^{x} P^1_{N-1,\alpha,\beta,\gamma,\delta}\,\partial_y v(y)\,dy,$$
$$\phi(x) = \phi^*(x) - \tfrac{1}{2}\phi^*(1)(x+1).$$

Clearly $\phi \in \mathscr{P}_N^0$. By the projection theorem,

$$\begin{aligned}\|P_{N,\alpha,\beta,\gamma,\delta}^{1,0} v - v\|_{1,\alpha,\beta,\gamma,\delta} &\leq \|\phi - v\|_{1,\alpha,\beta,\gamma,\delta} \\ &\leq c\|P_{N-1,\alpha,\beta,\gamma,\delta}^1 \partial_x v - \partial_x v\|_{\chi^{(\alpha,\beta)}} \\ &\quad + c|\phi^*(1)| + \|\phi - v\|_{\chi^{(\gamma,\delta)}}.\end{aligned}$$

Since $\gamma, \delta < 1$,

$$\begin{aligned}|\phi^*(1)| &= |v(1) - \phi^*(1)| \\ &\leq \left| \int_\Lambda \left(P_{N-1,\alpha,\beta,\gamma,\delta}^1 \partial_x v(x) - \partial_x v(x) \right) dx \right| \\ &\leq c\|P_{N-1,\alpha,\beta,\gamma,\delta}^1 \partial_x v - \partial_x v\|_{\chi^{(\gamma,\delta)}}.\end{aligned}$$

On the other hand,

$$\|\phi - v\|_{\chi^{(\gamma,\delta)}} \leq c\|P_{N-1,\alpha,\beta,\gamma,\delta}^1 \partial_x v - \partial_x v\|_{\chi^{(\gamma,\delta)}} + c|\phi^*(1)|.$$

Finally by virtue of Theorem 2.5,

$$\begin{aligned}\|P_{N,\alpha,\beta,\gamma,\delta}^1 v - v\|_{1,\alpha,\beta,\gamma,\delta} &\leq cN^{1-r}\|\partial_x v\|_{r-1,\chi^{(\alpha,\beta)},*} \\ &\leq cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*,2}.\end{aligned}$$

Next, assume that (2.17) holds and $\alpha, \beta > 0$. Let $g \in L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$ and consider the problem

$$a_{\alpha,\beta,\gamma,\delta}(w,z) = (g,z)_{\chi^{(\gamma,\delta)}}, \qquad \forall z \in H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda).$$

Clearly $\partial_x w(x)\chi^{(\alpha,\beta)}(x) \to 0$, as $|x| \to 1$. Moreover (2.22) holds. By an argument as in the proof of Theorem 2.5, we deduce the second result.

Another orthogonal projection is also used in the Jacobi spectral method. Let

$$\tilde{a}_{\alpha,\beta}(u,v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}}, \qquad \forall u, v \in H^1_{\chi^{(\alpha,\beta)}}(\Lambda). \tag{2.29}$$

The orthogonal projection $\tilde{P}^1_{N,\alpha,\beta}$: $H^1_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathscr{P}_N$ is such a mapping that for any $v \in H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$, $\tilde{P}^1_{N,\alpha,\beta}v(0) = v(0)$ and

$$\tilde{a}_{\alpha,\beta}\left(\tilde{P}^1_{N,\alpha,\beta}v - v, \phi\right) = 0, \qquad \forall \phi \in \mathscr{P}_N. \tag{2.30}$$

It is easy to prove the following result.

THEOREM 2.8. *For any* $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ *with* $r \geq 1$,

$$\|\tilde{P}^1_{N,\alpha,\beta} v - v\|_{1,\chi^{(\alpha,\beta)}} \leq cN^{1-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

The orthogonal projection $\tilde{P}^{1,0}_{N,\alpha,\beta}$: $H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \to \mathscr{P}^0_N$ *is such a mapping that for any* $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$,

$$\tilde{a}_{\alpha,\beta}\left(\tilde{P}^{1,0}_{N,\alpha,\beta} v - v, \phi\right) = 0, \qquad \forall \phi \in \mathscr{P}^0_N.$$

THEOREM 2.9. *If* (2.25) *or* (2.26) *holds and* $\alpha, \beta < 1$, *then for any* $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ *with* $r \geq 1$,

$$\|\tilde{P}^{1,0}_{N,\alpha,\beta} v - v\|_{1,\chi^{(\alpha,\beta)}} \leq cN^{1-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

Proof. Let

$$\phi^*(x) = \int_{-1}^{x} P_{N-1,\alpha,\beta}\, \partial_y v(y)\, dy, \qquad \phi(x) = \phi^*(x) - \tfrac{1}{2}\phi^*(1)(x+1).$$

By the projection theorem, Lemma 2.4, and Remark 2.3,

$$\|\tilde{P}^{1,0}_{N,\alpha,\beta} v - v\|_{1,\chi^{(\alpha,\beta)}} \leq c\|P_{N-1,\alpha,\beta}\, \partial_x v - \partial_x v\|_{\chi^{(\alpha,\beta)}} + c|\phi^*(1)|.$$

Since $\alpha, \beta < 1$, we have from Theorem 2.3 that

$$\begin{aligned} |\phi^*(1)| = |\phi^*(1) - v(1)| &\leq c\|P_{N-1,\alpha,\beta}\, \partial_x v - \partial_x v\|_{\chi^{(\alpha,\beta)}} \\ &\leq cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \end{aligned}$$

Remark 2.5. The second result of Theorem 2.9 is also valid for $\alpha = \beta = 0$. For $\alpha = \beta = -\frac{1}{2}$, the same result is valid for another $H^1_{0,\chi^{(\alpha,\beta)}}$-projection.

Another kind of approximation result exists.

THEOREM 2.10. *For any* $v \in H^r_{\chi^{(\alpha-1,\beta-1)}}(\Lambda)$ *with* $r \geq 0$,

$$\|\tilde{P}^1_{N,\alpha,\beta} v - v\|_{\chi^{(\alpha-1,\beta-1)}} \leq cN^{-r}\|v\|_{r,\chi^{(\alpha-1,\beta-1)},A},$$

where $|\alpha| + |\beta| \neq 0$.

Proof. Let

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l^{(\alpha-1,\beta-1)} J_l^{(\alpha-1,\beta-1)}(x),$$

$$\tilde{P}^1_{N,\alpha,\beta} v(x) = \sum_{l=0}^{N} a_l J_l^{(\alpha-1,\beta-1)}(x).$$

Take $\phi(x) = J_m^{(\alpha,\beta)}(x)$, $0 \le m \le N$, in (2.30). By (2.1),

$$\begin{aligned}
&\sum_{l=0}^{N} \left(a_l - \hat{v}_l^{(\alpha-1,\beta-1)}\right)\left(\partial_x J_l^{(\alpha-1,\beta-1)}, \partial_x J_m^{(\alpha-1,\beta-1)}\right)_{\chi^{(\alpha,\beta)}} \\
&\qquad - \sum_{l=N+1}^{\infty} \hat{v}_l^{(\alpha-1,\beta-1)}\left(\partial_x J_l^{(\alpha-1,\beta-1)}, \partial_x J_m^{(\alpha-1,\beta-1)}\right)_{\chi^{(\alpha,\beta)}} \\
&\qquad = \sum_{l=0}^{N} \lambda_l^{(\alpha-1,\beta-1)}\left(a_l - \hat{v}_l^{(\alpha-1,\beta-1)}\right)\left(J_l^{(\alpha-1,\beta-1)}, J_m^{(\alpha-1,\beta-1)}\right)_{\chi^{(\alpha-1,\beta-1)}} \\
&\qquad = \lambda_m^{(\alpha-1,\beta-1)} \gamma_m^{(\alpha-1,\beta-1)}\left(a_m - \hat{v}_m^{(\alpha-1,\beta-1)}\right) = 0.
\end{aligned}$$

Thus $a_l = \hat{v}_l^{(\alpha-1,\beta-1)}$, $1 \le l \le N$. Since $\tilde{P}^1_{N,\alpha,\beta} v(0) = v(0)$, $\tilde{P}^1_{N,\alpha,\beta}$ is exactly the same as $P_{N,\alpha-1\,\beta-1}$. Finally we complete the proof by using Theorem 2.3.

In applications of Jacobi approximation to nonlinear problems, we need to estimate the $W^{p,\infty}(\Lambda)$-norms of various orthogonal projections. Some of them are stated in the following.

THEOREM 2.11. *If* (2.15) *holds and* $\gamma, \delta \le 0$, *then for any* $v \in H^{1+d}_{\chi^{(\alpha,\beta)},*}(\Lambda) \cap H^d(\Lambda)$ *with* $d > 1$,

$$\|P^1_{N,\alpha,\beta,\gamma,\delta} v\|_\infty \le c\left(\|v\|_{1+d,\chi^{(\alpha,\beta)},*} + \|v\|_d\right).$$

If, in addtion, (2.17) *holds, then*

$$\|P^1_{N,\alpha,\beta,\gamma,\delta} v\|_\infty \le c\left(\|v\|_{d,\chi^{(\alpha,\beta)},*} + \|v\|_d\right).$$

Proof. By the imbedding theorem,

$$\begin{aligned}
\|P^1_{N,\alpha,\beta,\gamma,\delta} v\|_\infty &\le \|v\|_\infty + \|P^1_{N,\alpha,\beta,\gamma,\delta} v - v\|_{d/2} \\
&\le \|v\|_\infty + \|P^1_{N,\alpha,\beta,\gamma,\delta} v - P_{N,0,0} v\|_{d/2} + \|P_{N,0,0} v - v\|_{d/2}.
\end{aligned} \tag{2.31}$$

By Theorem 2.5 and an inverse inequality in $\mathscr{P}_N$,

$$\begin{aligned}\|P^1_{N,\alpha,\beta,\gamma,\delta}v - P_{N,0,0}v\|_{d/2} &\le cN^d\big(\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\| + \|P_{N,0,0}v - v\|\big)\\ &\le cN^d\big(\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{\chi^{(\gamma,\delta)}} + \|P_{N,0,0}v - v\|\big),\\ &\le c\big(\|v\|_{1+d,\chi^{(\alpha,\beta)},*} + \|v\|_d\big). \qquad (2.32)\end{aligned}$$

According to the property of the Legendre approximation,

$$\|P_{N,0,0}v - v\|_{d/2} \le c\|v\|_{(3/4)d}. \qquad (2.33)$$

Then the first result comes immediately.

If, in addition, (2.17) holds, then we have from Theorem 2.5 that

$$\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\| \le cN^{-d}\|v\|_{d,\chi^{(\alpha,\beta)},*}$$

and so

$$\|P^1_{N,\alpha,\beta,\gamma,\delta}v\|_\infty \le c\big(\|v\|_{d,\chi^{(\alpha,\beta)},*} + \|v\|_d\big).$$

THEOREM 2.12. *If* (2.25) *holds and* $\gamma, \delta \le 0$, *then for any* $v \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^{1+d}_{\chi^{(\alpha,\beta)},*}(\Lambda) \cap H^d(\Lambda)$ *with* $d > 1$,

$$\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v\|_\infty \le c\big(\|v\|_{1+d,\chi^{(\alpha,\beta)},*} + \|v\|_d\big).$$

If, in addition, (2.17) *holds, then*

$$\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v\|_\infty \le c\big(\|v\|_{d,\chi^{(\alpha,\beta)},*} + \|v\|_d\big).$$

Proof. By virtue of Theorem 2.6 and an argument as in the derivations of (2.31)–(2.33), we obtain

$$\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v\|_\infty \le \|v\|_\infty + \|{}_0P_{N,\alpha,\beta,\gamma,\delta}v - P_{N,0,0}v\|_{d/2} + \|P_{N,0,0}v - v\|_{d/2}$$

and

$$\begin{aligned}\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v - P_{N,0,0}v\|_{d/2} &\le cN^d\big(\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{\chi^{(\gamma,\delta)}} + \|P_{N,0,0}v - v\|\big)\\ &\le c\big(\|v\|_{1+d,\chi^{(\alpha,\beta)},*} + \|v\|_d\big).\end{aligned}$$

If, in addition, (2.17) holds, then

$$\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{\chi^{(\gamma,\delta)}} \le cN^{-d}\big(\|v\|_{d,\chi^{(\alpha,\beta)},*} + \|v\|_d\big).$$

Thus the proof is complete.

Remark 2.6. If (2.26) holds and $\gamma, \delta \le 0$, then for any $v \in {}^0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^{1+d}_{\chi^{(\alpha,\beta)},*}(\Lambda) \cap H^d(\Lambda)$ with $d > 1$, the norm $\|{}^0P^1_{N,\alpha,\beta,\gamma,\delta}v\|_\infty$ has the same bound as in Theorem 2.12.

THEOREM 2.13. *If $\gamma \le \alpha \le \gamma + 1$, $\delta \le \beta \le \delta + 1$ and $\gamma, \delta \le 0$, then for any $v \in H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^{1+d}_{\chi^{(\alpha,\beta)},*,2}(\Lambda) \cap H^d(\Lambda)$ with $d > 1$,*

$$\|P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v\|_\infty \le c\big(\|v\|_{1+d,\chi^{(\alpha,\beta)},*,2} + \|v\|_d\big).$$

Proof. By virtue of Theorem 2.7 and an argument as in the proof of Theorem 2.11, we reach the desired result.

Another kind of estimation exists.

THEOREM 2.14. *Let $\alpha \le -\gamma$, $\beta \le -\delta$. If (2.25) holds, then for any $v \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$,*

$$\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v\|_\infty \le 2\|v\|_{1,\alpha,\beta,\gamma,\delta}.$$

If (2.26) holds, then for any $v \in {}^0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$,

$$\|{}^0P^1_{N,\alpha,\beta,\gamma,\delta}v\|_\infty \le 2\|v\|_{1,\alpha,\beta,\gamma,\delta} + .$$

If $\gamma \le \alpha \le \gamma + 1$, $\delta \le \beta \le \delta + 1$ and $\gamma, \delta < 1$, then for any $v \in H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda)$,

$$\|P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v\|_\infty \le 2\|v\|_{1,\alpha,\beta,\gamma,\delta}.$$

If (2.25) or (2.26) holds and $\alpha, \beta < 1$, then for any $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$,

$$\|\tilde{P}^{1,0}_{N,\alpha,\beta}v\|_\infty \le 2\|v\|_{1,\chi^{(\alpha,\beta)}}.$$

Proof. We have from Theorem 2.6 that

$$
\begin{aligned}
&\left({}_0P^1_{N,\alpha,\beta,\gamma,\delta}v(x)\right)^2\\
&\quad= 2\int_{-1}^{x} {}_0P^1_{N,\alpha,\beta,\gamma,\delta}v(x)\ \partial_x\left({}_0P^1_{N,\alpha,\beta,\gamma,\delta}v(x)\right) dx\\
&\quad\le 2\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v\|_{\chi^{(-\alpha,-\beta)}}|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v|_{1,\chi^{(\alpha,\beta)}}\\
&\quad\le 2\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v\|_{\chi^{(\gamma,\delta)}}|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v|_{1,\chi^{(\alpha,\beta)}}\\
&\quad\le 2\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v\|_{1,\alpha,\beta,\gamma,\delta}\\
&\quad\le 2\|v\|^2_{1,\alpha,\beta,\gamma,\delta}.
\end{aligned}
$$

The remaining parts of this theorem can be proved similarly.

3. APPLICATIONS

This section is devoted to the applications of Jacobi approximation to singular problems. We first consider

$$-\partial_x\left(k(x)\ \partial_x U(x)\right) + b(x)U(x) = f(x), \qquad x \in \Lambda, \tag{3.1}$$

where $k(x) \ge 0$, $b(x) \ge 0$, and $f(x)$ are given functions. Assume that $k(x)$ and $b(x)$ degenerate as $|x| \to 1$. Without any loss of generality, suppose that $k(x) \sim \chi^{(\alpha,\beta)}(x)$, $b(x) \sim \chi^{(\gamma,\delta)}(x)$ as $|x| \to 1$, and that for certain positive constants c_1 and c_2, $\chi^{(\alpha,\beta)}(x) \le k(x) \le c_1\chi^{(\alpha,\beta)}(x)$, $\chi^{(\gamma,\delta)}(x) \le b(x) \le c_2\chi^{(\gamma,\delta)}(x)$. We look for the solution of (3.1) such that at least $k(x)\,\partial_x U(x) \to 0$ as $|x| \to 1$. A weak formulation of (3.1) is to find $U \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$ such that

$$(\partial_x U, \partial_x v)_k + (bU, v) = (f, v), \qquad \forall v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda). \tag{3.2}$$

If $f \in (H^1_{\alpha,\beta,\gamma,\delta}(\Lambda))'$, then (3.2) has a unique solution.

Let $U_N \in \mathscr{P}_N$ be the approximation to U, satisfying

$$
\begin{aligned}
&a_{\alpha,\beta,\gamma,\delta}(u_N,\phi) + \left((k-\chi^{(\alpha,\beta)})\ \partial_x u_N, \partial_x\phi\right)\\
&\qquad + \left((b-\chi^{(\gamma,\delta)})u_N,\phi\right) = (f,\phi), \qquad \forall\phi \in \mathscr{P}_N.
\end{aligned}\tag{3.3}
$$

For error estimate, let $U_N = P^1_{N,\alpha,\beta,\gamma,\delta}U$. By (3.2),

$$
\begin{aligned}
&a_{\alpha,\beta,\gamma,\delta}(U_N,\phi) + \left((k-\chi^{(\alpha,\beta)})\ \partial_x U, \partial_x\phi\right) + \left((b-\chi^{(\gamma,\delta)})U,\phi\right)\\
&\quad= (f,\phi), \qquad \forall\phi \in \mathscr{P}_N.
\end{aligned}\tag{3.4}
$$

Further let $\tilde{U}_N = u_N - U_N$. Then by (3.3) and (3.4),

$$a_{\alpha,\beta,\gamma,\delta}\left(\tilde{U}_N, \phi\right) + \left(\left(k - \chi^{(\alpha,\beta)}\right) \partial_x \tilde{U}_N, \partial_x \phi\right) + \left(\left(b - \chi^{(\gamma,\delta)}\right) U_N, \phi\right)$$
$$= F(\phi), \tag{3.5}$$

where

$$F(\phi) = \left(\left(k - \chi^{(\alpha,\beta)}\right) \partial_x (U - U_N), \partial_x \phi\right) + \left(\left(b - \chi^{(\gamma,\delta)}\right)(U - U_N), \phi\right).$$

Taking $\phi = \tilde{U}_N$ in (3.5), we get

$$\|\tilde{U}_N\|^2_{1,\alpha,\beta,\gamma,\delta} \le \left|F\left(\tilde{U}_N\right)\right|.$$

If (2.15) holds, then by Theorem 2.5,

$$\left|F\left(\tilde{U}_N\right)\right| \le \varepsilon \|\tilde{U}_N\|^2_{1,\alpha,\beta,\gamma,\delta} + \frac{c}{\varepsilon} N^{2-2r} \|U\|^2_{r,\chi^{(\alpha,\beta)},*}, \qquad \varepsilon > 0.$$

If, in addition, (2.17) holds and $k(x) \equiv \chi^{(\alpha,\beta)}(x)$, then

$$\left|F\left(\tilde{U}_N\right)\right| \le \varepsilon \|\tilde{U}_N\|^2_{1,\alpha,\beta,\gamma,\delta} + \frac{c}{\varepsilon} N^{-2r} \|U\|^2_{r,\chi^{(\alpha,\beta)},*}.$$

THEOREM 3.1. *Let* (2.15) *hold. If* $U \in H_{\chi^{(\alpha,\beta)},*}(\Lambda)$ *with* $r \ge 1$, *then*

$$\|\tilde{U}_N\|_{1,\alpha,\beta,\gamma,\delta} \le c N^{1-r} \|U\|_{r,\alpha,\beta,\gamma,\delta,*}.$$

If, in addition, (2.17) *holds and* $k(x) \equiv \chi^{(\alpha,\beta)}(x)$, *then for all* $0 \le \mu \le 1$,

$$\|U - u_N\|_{\mu,\alpha,\beta,\gamma,\delta} \le c N^{\mu - r} \|U\|_{r,\alpha,\beta,\gamma,\delta,*}.$$

Remark 3.1. If $k(x)$ degenerates at several distinct pionts, then we decompose the interval to several subintervals. Their extreme points coincide with those distinct points. Further we use different Jacobi approximations in different subintervals.

We next consider the logistic equation governing the population of budworms in an unbounded forest, say $\tilde{\Lambda} = \{y \mid 0 < y < \infty\}$. Suppose that the boundary condition at $y = -1$ is lethal, and the population $V(y,t)$ grows infinitely as $y \to \infty$, but at least $e^{-2y} \partial_y V(y,t) \to 0$. This problem is of the form

$$\begin{aligned} \partial_t V(y,t) - \partial_y^2 V(y,t) &= V(y,t)(1 - V(y,t)), \qquad y \in \tilde{\Lambda}, 0 < t \le T, \\ V(0,t) &= \lim_{y\to\infty} e^{-2y} \partial_y V(y,t) = 0, \qquad 0 \le t \le T, \\ V(y,0) &= V_0(y), \qquad y \in \tilde{\Lambda}. \end{aligned} \tag{3.6}$$

Now we make the variable transformation (see Guo [26])

$$y(x) = -2\ln(1-x) + 2\ln 2.$$

Obviously $y(-1) = 0$, $y(1) = \infty$ and for $x \in \Lambda$, $\frac{dx}{dy} = \frac{1}{2}(1-x) > 0$. Let $U(x,t) = V(y(x),t)$ and $U_0(x) = V_0(y(x))$. Then (3.6) becomes

$$\begin{aligned} \partial_t U(x,t) - \tfrac{1}{4}(1-x)\,\partial_x((1-x)\,\partial_x U(x,t)) \\ = U(x,t)(1-U(x,t)), \qquad x \in \Lambda, 0 < t \le T, \\ U(-1,t) = \lim_{x\to 1}(1-x)^2\,\partial_x U(x,t) = 0, \qquad 0 \le t \le T, \\ U(x,0) = U_0(x). \end{aligned} \tag{3.7}$$

A weak formulation of (3.7) is to find $U \in L^\infty(0,T; L^2(\Lambda)) \cap L^2(0,T; {}_0H^1_{\chi^{(2,0)}}(\Lambda))$ such that

$$\begin{aligned} \left(\partial_t U(x,t), v\right) + \tfrac{1}{4}a_{2,0,0,0}(U(t),v) + \tfrac{1}{4}\left(U(t), \partial_x v\right)_{\chi^{(1,0)}} \\ = \left(\tfrac{3}{2}U(t) - U^2(t), v\right), \qquad \forall v \in {}_0H^1_{2,0,0,0}(\Lambda), 0 < t \le T, \\ U(0) = U_0. \end{aligned} \tag{3.8}$$

If $U_0 \in {}_0H^1_{2,0,0,0}(\Lambda)$, then (3.8) has a unique solution.

Let $u_N(t)$ be the approximation to $U(t)$. The Jacobi spectral scheme for (3.8) is to find $u_N \in {}_0\mathscr{P}_N$ such that

$$\begin{aligned} \left(\partial_t u_N(t), \phi\right) + \tfrac{1}{4}a_{2,0,0,0}(u_N(t),\phi) + \tfrac{1}{4}\left(u_N(t), \partial_x \phi\right)_{\chi^{(1,0)}} \\ = \left(\tfrac{3}{2}u_N(t) - u_N^2(t), \phi\right), \qquad \forall \phi \in {}_0\mathscr{P}_N, 0 < t \le T. \end{aligned} \tag{3.9}$$

In addition, $u_N(0) = u_{N,0} = {}_0P^1_{N,2,0,0,0}U_0$.

We now analyze the stability of (3.9). Since it is a nonliear problem, it is not possible to possess the stability in the sense of Courant *et al.* [27]; also see Richtmeyer and Morton [28]. But it might be stable in the sense of Guo [29, 30]. To this end, assume that the initial value and computation in (3.9) have errors $\tilde{u}_{N,0}$ and $\tilde{f}$, respectively. They induce the error of u_N, denoted by $\tilde{u}_N$. Then we obtain from (3.9) that

$$\begin{aligned} \left(\partial_t \tilde{u}_N(t), \phi\right) + \tfrac{1}{4}a_{2,0,0,0}(\tilde{u}_N(t),\phi) + \tfrac{1}{4}\left(\tilde{u}_N(t), \partial_x \phi\right)_{\chi^{(1,0)}} \\ = F(t,\phi), \qquad \forall \phi \in {}_0\mathscr{P}_N, 0 < t \le T, \end{aligned} \tag{3.10}$$

where $\tilde{u}_N(0) = \tilde{u}_{N,0}$, and

$$F(t, \phi) = \left(\tfrac{3}{2}\tilde{u}_N(t) - 2u_N(t)\tilde{u}_N(t) - \tilde{u}_N^2(t) + \tilde{f}(t), \phi\right).$$

Take $\phi = \tilde{u}_N(t)$ in (3.10); we get

$$\partial_t \|\tilde{u}_N(t)\|^2 + \tfrac{1}{2}\|\tilde{u}(t)\|^2_{1,2,0,0,0} \leq 2\left|F(t, \tilde{u}_N(t))\right|. \tag{3.11}$$

By virtue of Theorem 2.1,

$$\|\tilde{u}_N(t)\|^3_{L^3} \leq cN\|\tilde{u}_N(t)\|^3. \tag{3.12}$$

Moreover, for any $\varepsilon > 0$,

$$\left|(\tilde{u}_N(t), \partial_x \tilde{u}_N(t))_{\chi^{(1,0)}}\right| \leq \varepsilon\|\tilde{u}_N(t)\|^2_{1,2,0,0,0} + \frac{1}{4\varepsilon}\|\tilde{u}_N(t)\|^2. \tag{3.13}$$

Let $|||\, v \,|||_\infty = \max_{0 \leq t \leq T}\|v(t)\|_\infty$, and

$$E(v, t) = \|v(t)\|^2 + \int_0^t \|v(s)\|^2_{1,2,0,0,0}\, ds,$$

$$\rho(v, w, t) = \|v\|^2 + \int_0^t \|w(s)\|^2\, ds.$$

By substituting (3.12) and (3.13) into (3.11) and integrating the resulting inequality, we get

$$E(\tilde{u}_N, t) \leq c(|||\, u_N \,|||_\infty + 1)\int_0^t \left(E(\tilde{u}_N, s) + NE^{3/2}(\tilde{u}_N, s)\right) ds + \rho\left(\tilde{u}_{N,0}, \tilde{f}, t\right). \tag{3.14}$$

Finally we get the following result.

THEOREM 3.2. *If $\rho(u_N, \tilde{f}, T) \leq b_1/N^2$, then for all $t \leq T$,*

$$E(\tilde{u}_N, t) \leq \rho\left(\tilde{u}_{N,0}, \tilde{f}, t\right) \leq e^{b_2 t},$$

where b_1 and b_2 are certian positive constants depending only on $|||\, u_N \,|||_\infty$.

Remark 3.2. In actual computation, the value of N is fixed. Theorem 3.2 indicates that if the average error of data does not exceed $b_1^{1/2}N^{-1}$, then the error of numerical solution is still majorized by it. It also tells us that for large N, we should pay more attention to the error of initial data and the error caused in computation.

We now deal with the convergence of (3.9). Let $U_N = {}_0P^1_{N,2,0,0,0}U$. By (3.8),

$$\big(\partial_t U_N(t), \phi\big) + \tfrac{1}{4}a_{2,0,0,0}\big(U_N(t), \phi\big) + \tfrac{1}{4}\big(U_N(t), \partial_x \phi\big)_{\chi^{(1,0)}}$$
$$= \big(\tfrac{3}{2}U_N(t) - U_N^2(t), \phi\big) + \sum_{j=1}^{4} G_j(t, \phi), \qquad \forall \phi \in {}_0\mathscr{P}_N, 0 < t \le T, \tag{3.15}$$

where

$$G_1(t, \phi) = \big(\partial_t U_N(t) - \partial_t U(t), \phi\big),$$
$$G_2(t, \phi) = \tfrac{3}{2}\big(U(t) - U_N(t), \phi\big),$$
$$G_3(t, \phi) = \big(U_N^2(t) - U^2(t), \phi\big),$$
$$G_4(t, \phi) = \tfrac{1}{4}\big(U_N(t) - U(t), \partial_x \phi\big)_{\chi^{(1,0)}}.$$

Furthermore let $\tilde{U}_N = u_N - U_N$. We obtain from (3.9) and (3.15) that

$$\big(\partial_t \tilde{U}_N(t), \phi\big) + \tfrac{1}{4}a_{2,0,0,0}\big(\tilde{U}_N(t), \phi\big) + \tfrac{1}{4}\big(\tilde{U}_N(t), \partial_x \phi\big)_{\chi^{(1,0)}}$$
$$= \big(\tfrac{3}{2}\tilde{U}_N(t) - 2U_N(t)\tilde{U}_N(t) - \tilde{U}_N^2(t), \phi\big)$$
$$- \sum_{j=1}^{4} G_j(t, \phi), \qquad \forall \phi \in {}_0\mathscr{P}_N, 0 < t \le T. \tag{3.16}$$

In addition $\tilde{U}_N(0) = 0$. Comparing (3.16) with (3.10), we can derive an estimate like (3.14). But U_N, $\tilde{U}_N$, and $|||\, u_N \,|||$ are now replaced by U_N, $\tilde{U}_N$, and $|||\, U_N \,|||$, respectively. Thus it remains to estimate $|G_j(t, \tilde{U}_N(t))|$, $1 \le j \le 3$. By Theorem 2.6,

$$\Big|G_1\big(t, \tilde{U}_N(t)\big)\Big| \le cN^{-2r}\| \partial_t U(t) \|^2_{r+1, \chi^{(2,0)}, *} + c\Big\| \tilde{U}_N(t) \Big\|^2,$$

$$\Big|G_2\big(t, \tilde{U}_N(t)\big)\Big| \le cN^{-2r}\| U(t) \|^2_{r+1, \chi^{(2,0)}, *} + c\Big\| \tilde{U}_N(t) \Big\|^2.$$

By Theorems 2.6 and 2.12, for $d > 1$,

$$\Big|G_3\big(t, \tilde{U}_N(t)\big)\Big|$$
$$\le cN^{-2r}\big(\| U(t) \|_d + \| U(t) \|_{1+d, \chi^{(2,0)}, *}\big)\| U(t) \|_{r+1, \chi^{(2,0)}, *} + c\Big\| \tilde{U}_N(t) \Big\|^2.$$

Theorem 2.6 also implies

$$\Big|G_4\big(t, \tilde{U}_N(t)\big)\Big| \le \frac{c}{\varepsilon}N^{-2r}\| U(t) \|^2_{r+1, \chi^{(2,0)}, *} + \varepsilon\Big\| \tilde{U}_N(t) \Big\|^2_{1,2,0,0,0}.$$

Finally we obtain the following conclusion.

THEOREM 3.3. *If for $r, d > 1$,*

$$U \in L^2\left(0, T; H^{r+1}_{\chi^{(2,0)},*}(\Lambda)\right) \cap L^\infty\left(0, T; H^d(\Lambda) \cap {}_0H^1_{\chi^{(2,0)},*}(\Lambda) \cap H^{1+d}_{\chi^{(2,0)},*}(\Lambda)\right),$$

then for all $0 \le t \le T$,

$$E(U - u_N, t) \le b_3 N^{-2r},$$

where b_3 is a positive constant depending only on the norms of U in the spaces mentioned.

Remark 3.3. In actual computation, we need to discretize the term $\partial_t U_N(t)$ in (3.9). If we use forward difference, then we require some conditions on the product τN^2, τ being the step size in time. According to a result similar to Theorem 2.2, τN^2 must be bounded by some positive constant.

ACKNOWLEDGMENTS

The author thanks Professor Ma He-ping and Professor C. W. Shu for their important suggestions.

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